Chapter 7

Brownian Motion: Fokker-Planck Equation

The Fokker-Planck equation is the equation governing the time evolution of the probability density of the Brownian particla. It is a second order differential equation and is exact for the case when the noise acting on the Brownian particle is Gaussian white noise. A general Fokker-Planck equation can be derived from the Chapman-Kolmogorov equation, but we also like to find the Fokker-Planck equation corresponding to the time dependence given by a Langevin equation.

The derivation of the Fokker-Planck equation is a two step process. We first derive the equation of motion for the probability density 4/varrho(x, v, t)4 to find the Brownian particle in the interval $(x, x+\mathrm{d}x)$ and $(v, v+\mathrm{d}v)$ at time t for one realization of the random force $\xi(t)$. We then obtain an equation for

$$P(x, v, t) = \langle \varrho(x, v, t) \rangle_\xi$$

i.e. the average of $\varrho(x, v, t)$ over many realizations of the random force. The probability density $P(x, v, t)$ is the macroscopically observed probability density for the Brownian particel.

7.1 Probability flow in phase-space

Let us obtain the probability to find the Brownian particle in the interval $(x, x + \mathrm{d}x$ and $(v, v + \mathrm{d}v)$ at time t. We will consider the space of coordiantes $\boldsymbol{x} = (x, v)$. The Brownian particle is located in the infinitesimal ara $\mathrm{d}x\mathrm{d}v$ with probablity $\varrho(x, v, t)\mathrm{d}x\mathrm{d}v$. The velocity of the particle at point (x, v) is given by $\dot{\boldsymbol{x}} = (\dot{x}, \dot{v})$ and the current density is $\dot{\boldsymbol{x}}\varrho$. Since the Brownian particle must lie somewhere in the phase-space $-\infty < x < \infty, _\infty < v < \infty$ we have the condition

$$\int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}v \varrho(x, v, t) = 1$$

Let us now consider a finite area, or volume, V_0 in this space. Since the Brownian particle cannot be destroyed a change in the probability contained in V_0 must be due to a flow of probability through the surface S_0 surrounding V_0. Thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\int_{V_0}\mathrm{d}x\mathrm{d}v\varrho(x,v,t) = -\int_{S_0}\varrho(x,v,t)\dot{\boldsymbol{x}}\cdot\mathrm{d}\boldsymbol{S}$$

We can now use Gauss theorem to change the surface integral into a volume integral.

$$\int\int_{V_0}\mathrm{d}x\mathrm{d}v\frac{\partial}{\partial t}\varrho(x,v,t) = -\int\int_{V_0}\mathrm{d}x\mathrm{d}v\nabla\cdot(\dot{\boldsymbol{x}}\varrho(x,v,t))$$

Since V_0 is fixed and arbitrary we find the continuity equation

$$\frac{\partial}{\partial t}\varrho(x,v,t) = -\nabla\cdot(\dot{\boldsymbol{x}}\varrho(x,v,t)) = -\frac{\partial}{\partial x}\left(\dot{x}\varrho(x,v,t)\right) - \frac{\partial}{\partial v}\left(\dot{v}\varrho(x,v,t)\right) \tag{7.1}$$

This is the continuity equation in phase-space which just state that probability is conserved.

7.2 Probability flow for Brownian particle

In order to write (7.1) explicitly for a Brownian particle we must know the Langevin equation governing the evolution of the particle. For a particle moving in the presence of a potnetial $V(x)$ the Langevin equations are

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= v\\ \frac{\mathrm{d}v}{\mathrm{d}t} &= -\frac{\gamma}{m}v + \frac{1}{m}F(x) + \frac{1}{m}\xi(t)\end{aligned} \tag{7.2}$$

where the force $F(x) = -V'(x)$. Inserting (7.2) into (7.1) gives

$$\begin{aligned}\frac{\partial}{\partial t}\varrho(x,v,t) &= -\frac{\partial}{\partial x}\left(v\varrho(x,v,t)\right) + \frac{\gamma}{m}\frac{\partial}{\partial v}\left(v\varrho(x,v,t)\right) - \frac{1}{m}F(x)\frac{\partial}{\partial v}\varrho(x,v,t)\\ &- \frac{1}{m}\xi(t)\frac{\partial}{\partial v}\varrho(x,v,t) = -L_0\varrho(x,v,t) - L_1(t)\varrho(x,v,t)\end{aligned}$$

where the differential operators L_0 and L_1 are defined as

$$\begin{aligned}L_0 &= v\frac{\partial}{\partial x} - \frac{\gamma}{m} - \frac{\gamma}{m}v\frac{\partial}{\partial v} + \frac{1}{m}F(x)\frac{\partial}{\partial v}\\ L_1 &= \frac{1}{m}\xi(t)\frac{\partial}{\partial v}\end{aligned}$$

Since $\xi(t)$ is a stochastic variable the time evolution of ϱ will be different for each realization of $\xi(t)$. However when we observe an actual Brownian particle we are observing the average effect of the random force on it. Therefore we introduce an observable probability density $P(x,v,t) = \langle\xi(t)\rangle_\xi$.

Let

$$\varrho(t) = \mathrm{e}^{-L_0 t}\sigma(t)$$

then

$$\frac{\partial}{\partial t}\sigma(x,v,t) = -\mathrm{e}^{-L_0 t}L_1(t)\mathrm{e}^{-L_0 t}\sigma(x,v,t) = -V(t)\sigma(x,v,t)$$

This equation has the formal solution

$$\sigma(t) = \exp\left[-\int_0^t \mathrm{d}t' V(t')\right]\sigma(0)$$

which follows since formally

$$\begin{aligned}
\sigma(t) &= \sigma(0) - \int_0^t \mathrm{d}t_1 V(t_1)\sigma(t_1) = \sigma(0) - \int_0^t \mathrm{d}t_1 V(t_1)\sigma(0) + \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 V(t_1)V(t_2)\sigma(0) + \ldots \\
&+ (-1)^n \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \cdots \int_0^{t_n} \mathrm{d}t_{n-1} V(t_1)V(t_2)\ldots V(t_{n-1})\sigma(0) + \ldots \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\left[\int_0^t \mathrm{d}t_1 V(t_1)\right]^n \sigma(0) = \exp\left[-\int_0^t \mathrm{d}t' V(t')\right]\sigma(0)
\end{aligned}$$

The third step follows since by changing the order of integration and then varibles

$$\int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 V(t_1)V(t_2) = \int_0^t \mathrm{d}t_2 \int_{t_2}^t \mathrm{d}t_1 V(t_1)V(t_2) = \int_0^t \mathrm{d}t_1 \int_{t_1}^t \mathrm{d}t_2 V(t_1)V(t_2)$$

so that

$$\int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 V(t_1)V(t_2) = \frac{1}{2}\left[\int_0^t \mathrm{d}t_1 V(t_1)\right]^2$$

Also assume that

$$\int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \cdots \int_0^{t_n} \mathrm{d}t_{n-1} V(t_1)V(t_2)\ldots V(t_{n-1}) = \frac{1}{n!}\left[\int_0^t \mathrm{d}t_1 V(t_1)\right]^n \tag{7.3}$$

then by taking the derivative

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \cdots \int_0^{t_{n+1}} \mathrm{d}t_n V(t_1)V(t_2)\ldots V(t_n) \\
&= V(t)\int_0^t \mathrm{d}t_2 \int_0^{t_2} \mathrm{d}t_3 \cdots \int_0^{t_n} \mathrm{d}t_{n+1} V(t_1)V(t_2)\ldots V(t_n) \\
&= V(t)\frac{1}{n!}\left[\int_0^t \mathrm{d}t_1 V(t_1)\right]^n = \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{(n+1)!}\left[\int_0^t \mathrm{d}t_1 V(t_1)\right]^{n+1}
\end{aligned}$$

By induction it therefore follows that (7.3) holds. Taking the average $\langle\cdots\rangle_\xi$ over the Gaussian noice $\xi(t)$ we see that $\langle\sigma(t)\rangle_\xi$ is the characteristic function of the random variable $X(t) = \mathrm{i}\int_0^t \mathrm{d}t_1 V(t_1)$. This must again be a Gaussian variable with $\langle X(t)\rangle_\xi = 0$ and the variance is

$$\langle X(t)^2\rangle = \frac{1}{2}\int_0^t \mathrm{d}t_1 \int_0^t \mathrm{d}t_2 \langle V(t_1)V(t_2)\rangle$$

Since the characteristic function for the Gaussian variable $X(t)$ is $\exp\left(\mathrm{i}X(t)\right) = \exp\left(\mathrm{i}\mu_X - \langle X(t)^2\rangle/2\right)$ we find

$$\langle\sigma(t)\rangle_\xi = \exp\left(\frac{1}{2}\int_0^t \mathrm{d}t_1 \int_0^t \mathrm{d}t_2 \langle V(t_1)V(t_2)\rangle\right)\sigma(0) \tag{7.4}$$

This formula is just a special case of a cumulant expansion. The integral in (7.4) becomes

$$\begin{aligned}\frac{1}{2}\int_0^t \mathrm{d}t_1 \int_0^t \mathrm{d}t_2 \langle V(t_1)V(t_2)\rangle_\xi &= \frac{1}{2}\int_0^t \mathrm{d}t_1 \int_0^t \mathrm{d}t_2 \langle \mathrm{e}^{L_0 t_1} \frac{1}{m}\xi(t_1)\frac{\partial}{\partial v}\mathrm{e}^{-L_0 t_1}\mathrm{e}^{L_0 t_2}\frac{1}{m}\xi(t_2)\frac{\partial}{\partial v}\mathrm{e}^{-L_0 t_2}\rangle\xi \\ &= \frac{g}{2m^2}\int_0^t \mathrm{d}t_1 \mathrm{e}^{L_0 t_1}\frac{\partial^2}{\partial v^2}\mathrm{e}^{-L_0 t_1}\end{aligned}$$

Then

$$\frac{\partial}{\partial t}\langle \sigma(x,v,t)\rangle_\xi = \frac{g}{2m^2}\mathrm{e}^{L_0 t}\frac{\partial^2}{\partial v^2}\mathrm{e}^{-L_0 t}\langle \sigma(x,v,t)\rangle_\xi$$

This gives for $\langle \varrho(x,v,t)\rangle_\xi$

$$\frac{\partial}{\partial t}\langle \varrho(x,v,t)\rangle_\xi = -L_0\langle \varrho(x,v,t)\rangle_\xi + \frac{g}{2m^2}\frac{\partial^2}{\partial v^2}\langle \varrho(x,v,t)\rangle_\xi$$

and for the probability distribution

$$\frac{\partial}{\partial t}P(x,v,t) = -v\frac{\partial}{\partial x}P(x,v,t) - \frac{\partial}{\partial v}\left[\left(\frac{\gamma}{m}v - \frac{1}{m}F(x)\right)P(x,v,t)\right] + \frac{g}{2m^2}\frac{\partial^2}{\partial v^2}P(x,v,t) \tag{7.5}$$

This is the Fokker-Planck equation for the probability $P\mathrm{d}x\mathrm{d}v$ to find the Brownian particle in the interval $(x, x+\mathrm{d}x, (v, v+\mathrm{d}v)$ at time t.

We can write the Fokker-Planck equation as a continuity equation

$$\frac{\partial}{\partial t}P(x,v,t) = -\nabla\cdot\boldsymbol{j}$$

where $\nabla = \boldsymbol{e}_x\partial/\partial x + \boldsymbol{e}_v\partial/\partial v$ and the probability current is

$$\boldsymbol{j} = \boldsymbol{e}_x vP - \boldsymbol{e}_v\left[\left(\frac{\gamma}{m}v - \frac{1}{m}F(x)\right)P + \frac{g}{2m^2}\frac{\partial}{\partial v}P\right]$$

7.3 General Fokker-Planck equation

We can obtain the Fokker-Planck equation for a quite general Langevin equation for the dynamics of a set of fluctuating variables

$$\boldsymbol{a} = \{a_1, a_2, \ldots\}$$

We assume a general friction term $\nu_j(a_1, a_2, \ldots) = \nu_j(\boldsymbol{a})$ and assume a Gaussian noise $\xi_j(t)$ where

$$\begin{aligned}\langle \xi_j(t)\rangle &= 0 \\ \langle \xi_i(t_2)\xi_j(t_1)\rangle &= g_{ij}\delta(t_2 - t_{1)}\end{aligned}$$

The Langevin equation becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}a_j(t) = \nu_j(\boldsymbol{a}) + \xi_j(t)$$

or in vector form

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{a}(t) = \nu(\boldsymbol{a}) + \xi(t)$$

We ask for the probability distribution

$$P(\boldsymbol{a},t) = \langle \varrho(\boldsymbol{a},t)\rangle_\xi$$

Again from conservation of probability

$$\frac{\partial}{\partial t}\varrho(\boldsymbol{a},t) + \frac{\partial}{\partial \boldsymbol{a}}\cdot\left[\frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t}\varrho(\boldsymbol{a},t)\right] = 0$$

Usin the Langevin equation to solve for $\mathrm{d}\boldsymbol{a}/\mathrm{d}t$ we find

$$\frac{\partial}{\partial t}\varrho(\boldsymbol{a},t) = \frac{\partial}{\partial \boldsymbol{a}}\cdot(\nu(\boldsymbol{a})\varrho(\boldsymbol{a},t)) - \frac{\partial}{\partial \boldsymbol{a}}\cdot(\xi(t)\varrho(\boldsymbol{a},t)) = -\left[L_0 + L_1(t)\right]\varrho(\boldsymbol{a},t)$$

where

$$\begin{aligned} L_0 &= \left(\frac{\partial}{\partial \boldsymbol{a}}\cdot\nu(\boldsymbol{a})\right) + \nu(\boldsymbol{a})\cdot\frac{\partial}{\partial \boldsymbol{a}} \\ L_1(t) &= \xi(t)\cdot\frac{\partial}{\partial \boldsymbol{a}} \end{aligned}$$

Following the steps as above we find

$$\frac{\partial}{\partial t}P(\boldsymbol{a},t) = -\frac{\partial}{\partial \boldsymbol{a}}\cdot(\nu(\boldsymbol{a})P(\boldsymbol{a},t)) + \frac{1}{2}\frac{\partial}{\partial \boldsymbol{a}}\cdot\boldsymbol{g}\cdot\frac{\partial}{\partial \boldsymbol{a}}P(\boldsymbol{a},t) \tag{7.6}$$

Here $\boldsymbol{g}$ is a tensor with elements g_{ij}.

Example

Our previous result can be obtained as a special case of (7.6). The Langevin equations are

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= v \\ \frac{\mathrm{d}v}{\mathrm{d}t} &= -\frac{\gamma}{m}v + \frac{1}{m}F(x) + \frac{1}{m}\xi(t) \end{aligned}$$

where

$$\langle \xi(t_2)\xi(t_{1)}\rangle = 2\gamma k_{\mathrm{B}}T\delta(t_2 - t_1)$$

Then

$$\begin{aligned} \boldsymbol{a} &= \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \nu(\boldsymbol{a}) = \begin{pmatrix} v \\ -\frac{\gamma}{m}v + \frac{1}{m}F(x) \end{pmatrix} \\ \xi(t) &= \begin{pmatrix} 0 \\ \frac{1}{m}\xi(t) \end{pmatrix}, \qquad \boldsymbol{g} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{2\gamma k_{\mathrm{B}}T}{m^2} \end{pmatrix} \end{aligned}$$

The Fokker-Planck equation becomes

$$\frac{\partial}{\partial t}P(x,v,t) = -\frac{\partial}{\partial x}\left[vP(x,v,t)\right] - \frac{\partial}{\partial v}\left[\left(-\frac{\gamma}{m}v + \frac{1}{m}F(x)\right)P(x,v,t)\right] + \frac{\gamma k_{\mathrm{B}}T}{m^2}\frac{\partial^2}{\partial v^2}P(x,v,t)$$